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On the perturbed rotational motions of a rigid body in the Lagrange case with variable restoring moment

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Abstract : Perturbed rotational motions of a rigid body that are close to regular precession in the Lagrange case when the restoring moment is variable are investigated, depending on the Euler's angles and the components of angular velocity. Such restoring moment is introduced by making the rotation of the rigid body in an electro-magnetic field It is assumed that the angular velocity of the body is large, its direction is close to the axis of dynamic symmetry of the body, and that the three projections of the vector of the perturbing moment onto the principal axes of inertia of the body are small as compared to the restoring moment The average method is employed and the averaged system of equations of motion is solved in the first and second approximations Examples are considered. Numerical results are obtained, for the averaged system of equations of motion, and are discussed in details

Keywords : Perturbed rotational motion, rigid body, variable restoring moment

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1. Introduction

In the last few decades, considerable interest arose in the generalization of the classical problem of motion of a rigid body about a fixed point. The motion of charged rigid body in uniform gravitational and magnetic fields was also considered [1–4]. In [5,6], the restoring moment is taken as a function of nutation angle only.

In the present work, we get a restoring moment as a function of more than one variable, namely, the Euler's angles and the components of the angular velocity.

2. Formulation of the problem

We consider the motion of dynamically symmetrical rigid body about a fixed point 0 under the action of variable restoring moment and perturbing moments, the equations of motion can be written in the form :

$$
\begin{aligned}
A\dot{p} + (c-A)qr &= k\sin\theta\cos\phi + M_1 \\
A\dot{q} + (A-c)pr &= -k\sin\theta\sin\phi + M_2 \\
c\dot{r} &= M_3 \\
\dot{\psi} &= (p\sin\phi + q\cos\phi)\operatorname{cosec}\theta \\
\dot{\theta} &= p\cos\phi - q\sin\phi \\
\dot{\phi} &= r - (p\sin\phi + q\cos\phi)\operatorname{ctg}\theta
\end{aligned}
\tag{1}
$$

Dynamic equations (1) are written in projections onto the principal axes of inertia of the body, passing through the fixed point 0. Here p, q, r are the projections of the angular velocity vector of the body onto these axes; M_i (i = 1, 2, 3) are the projections of the vector of the perturbing moment onto these axes, which are 2π-periodic of the Euler angles ψ, θ, ϕ; A and c are the equatorial and axial moments of inertia of the body relative to the fixed point 0, $A \neq c$.

The perturbing moments M_i in (1) are assumed to be known functions of their arguments, when M_i (i = 1, 2, 3) = 0 and $k = mgL$ = const., then the system of equations (1) corresponds to the Lagrange case. Here m is the mass of the body; g is acceleration due to gravity; and L is the distance from fixed point 0 to the center of gravity of the body. Equations (1) may describe motions of a Lagrange top acted upon by perturbations of various physical origin, as well as motions of a free rigid body relative to the center of mass, when this body is acted upon by a restoring moment generated by aerodynamic forces, and certain perturbing moments.

We make the following initial assumptions :

$$
p^2 + q^2 \langle\langle\, r^2,\ cr^2 \,\rangle\rangle\, k,\ \left|M_i\right| \langle\langle\, k, \qquad (i = 1, 2, 3)
\tag{2}
$$

which means that the direction of the angular velocity of the body is close to the axis of dynamic symmetry; the angular velocity is large, so that the kinetic energy of the body is much greater than the potential energy resulting from the restoring moment; the three projections of the vector of perturbing moment onto the principal axes of inertia of the body are small as compared to the restoring moment. Inequalities (2) allow us to introduce the small parameter $\varepsilon << 1$ and to set

$$
\left.
\begin{aligned}
&p = \varepsilon P,\ q = \varepsilon Q,\ k = \varepsilon K \\
&M_i = \varepsilon^2 M_i^{*}\ (P, Q, r, \psi, \theta, \phi, t)\quad (i = 1, 2, 3)
\end{aligned}
\right]
\tag{3}
$$

The new variables P and Q, as well as the variables and constants r, ψ, t, θ, ϕ, K, A, c, M_i^{*} (i = 1, 2, 3) are assumed to be bounded quantities of order unity as $\varepsilon \to 0$.

The problem under investigation is the study of the asymptotic behaviour of the solution of the system (1) for small ε when the conditions (2) and (3) are satisfied. This will be done by employing the average method [7–9], which is extensively employed in problems of dynamic of rigid bodies in a time interval of order ε^{-1}. Moiseev [8], reported that this method was employed to investigate a variety of problems of dynamics, essentially for bodies with dynamic symmetry. Chernous'ko [10] was the first to perform averaging with respect to Euler-Poinsot motion for a symmetrical body. A number of workers [11–14], have investigated perturbed motions of a rigid body.

3. Treatment of average

Introducing the change of variables (3) into (1), we get :

$$\begin{aligned}
A\dot{P} + (c-A)Qr &= K\sin\theta\cos\phi + \varepsilon M_1^{*} \\
A\dot{Q} + (A-c)Pr &= -K\sin\theta\sin\phi + \varepsilon M_2^{*} \\
c\dot{r} &= \varepsilon^2 M_3 \\
\dot{\psi} &= \varepsilon(P\sin\phi + Q\cos\phi)\operatorname{cosec}\theta \\
\dot{\theta} &= \varepsilon(P\cos\phi - Q\sin\phi) \\
\dot{\phi} &= r - \varepsilon(P\sin\phi + Q\cos\phi)\operatorname{ctg}\theta
\end{aligned} \tag{4}$$

Let us consider the zero-approximation system; by putting $\varepsilon = 0$ in (4). The last four equations in (4) yield;

$$r = r_0,\quad \psi = \psi_0,\quad \theta = \theta_0,\quad \phi = r_0 t + \phi_0 \tag{5}$$

Here $r_0, \psi_0, \theta_0, \phi_0$ are constants equal to the initial values of the corresponding variables at $t = 0$ Substituting (5) into the first two equations of the system (4) for $\varepsilon = 0$, and integrating the resultant system of the two linear equations for P, Q. The solution is given in the form

$$\begin{aligned}
P &= a\cos\gamma_0 + b\sin\gamma_0 + \lambda_0\sin(r_0 t + \phi_0), \\
Q &= a\sin\gamma_0 - b\cos\gamma_0 + \lambda_0\cos(r_0 t + \phi_0), \\
a^{(0)} &= P_0 - \lambda_0\sin\theta_0, \\
b^{(0)} &= -Q_0 + \lambda_0\cos\phi_0,
\end{aligned} \tag{6}$$

such that $\quad \lambda_0 = K_0 c^{-1} r_0^{-1}\sin\theta_0, \qquad \gamma_0 = n_0 t$

$$n_0 = (c-A)A^{-1}r_0 \neq 0, \qquad \left|n_0 / r_0\right| \le 1, \qquad K_0 = K|_{t=0}$$

Here P_0, Q_0 are the initial values of the new variables P, Q, introduced in accordance with (3), while the variable $\gamma = \gamma_0$ has the meaning of the oscillation phase and is defined by the equation

$$\dot{\gamma} = n, \quad \gamma(0) = 0, \quad n = (c-A)A^{-1}r \tag{7}$$

For $\varepsilon = 0$ we have $\gamma = \gamma_0 = n_0 t$ which is consistant with (6). The two systems (5) and (6) define the general solution of the system (4) for $\varepsilon = 0$. By eliminating the constants with allowance for (5), it is possible to rewrite the first two expressions in (6) in the equivalent form :

$$\left.\begin{aligned} P &= a\cos\gamma + b\sin\gamma + \lambda\sin(\gamma+\phi), \\ Q &= a\sin\gamma - b\cos\gamma + \lambda\cos(\gamma+\phi), \end{aligned}\right] \tag{8}$$

and to solve for a, b

$$\begin{aligned} a &= P\cos\gamma + Q\sin\gamma - \lambda\sin(\gamma+\phi), \\ b &= P\sin\gamma - Q\cos\gamma + \lambda\cos(\gamma+\phi) \end{aligned} \tag{9}$$

System (4) is essentially nonlinear (the natural oscillation frequency of the variables P, Q depends on the slow variable r) and therefore an additional variable δ is introduced, which is defined by the equation

$$r = r_0 + \varepsilon\delta \tag{10}$$

Now, let us consider system (4) for $\varepsilon \neq 0$ and expressions (8) and (9) as change of variables formulae (that contain the variable γ). This defines a change from variables P, Q to variables a, b of the Van der pol type [7] and *vice versa*. Using these formulae is system (4), we convert from the variables P, Q, r, ψ, θ, ϕ, γ to the new variables a, b, r, ψ, θ, α, δ where

$$\alpha = \gamma + \phi \tag{11}$$

After some manipulations, a system of seven equations is obtained, which is more convenient for subsequent investigation, then the six equations in (4) will take the form :

$$\begin{aligned} \dot{a} = {} & \varepsilon A^{-1}[M_1^0\cos\gamma + M_2^0\sin\gamma] + A^{-1}K_0\sin\theta\cos\alpha - A^{-1}K_0\sin\theta\cos\alpha \\ & - \varepsilon K_0 c^{-1}r_0^{-1}\cos\theta(b - K_0c^{-1}r_0^{-1}\sin\theta\cos\alpha) - \varepsilon c^{-1}r_0^{-1}\sin\theta\sin\alpha\frac{\partial K_0}{\partial\theta} \\ & \times(a\cos\alpha + b\sin\alpha) + \varepsilon^2 K_0c^{-1}r_0^{-2}\delta\cos\theta(b - 2K_0c^{-1}r_0^{-1}\sin\theta\cos\alpha) \\ & + \varepsilon^2c^{-1}r_0^{-2}\delta\sin\theta\sin\alpha\frac{\partial K_0}{\partial\theta}(a\cos\alpha + b\sin\alpha) + \varepsilon^2K_0c^{-2}r_0^{-2}M_3^0\sin\theta\sin\alpha \\ & - \varepsilon^2c^{-2}r_0^{-1}M_3^0\frac{\partial K_0}{\partial r}\sin\theta\sin\alpha \end{aligned} \tag{12.1}$$

$$\begin{aligned} \dot{b} = {} & \varepsilon A^{-1}[M_1^0\sin\gamma - M_2^0\sin\gamma] - A^{-1}K_0\sin\theta\sin\alpha + A^{-1}K_0\sin\theta\sin\alpha \\ & + \varepsilon K_0c^{-1}r_0^{-1}\cos\theta(a + K_0c^{-1}r_0^{-1}\sin\theta\sin\alpha) + \varepsilon c^{-1}r_0^{-1}\sin\theta\cos\alpha\frac{\partial K_0}{\partial\theta} \\ & \times(a\cos\alpha + b\sin\alpha) - \varepsilon^2K_0c^{-1}r_0^{-2}\delta\cos\theta(a + 2K_0c^{-1}r_0^{-1}\sin\theta\sin\alpha) \\ & - \varepsilon^2c^{-1}r_0^{-2}\delta\sin\theta\cos\alpha\frac{\partial K_0}{\partial\theta}(a\cos\alpha + b\sin\alpha) - \varepsilon^2K_0c^{-2}r_0^{-2}M_3^0\sin\theta\cos\alpha \\ & + \varepsilon^2c^{-2}r_0^{-1}M_3^0\frac{\partial K_0}{\partial r}\sin\theta\cos\alpha \end{aligned} \tag{12.2}$$

$$\dot{\delta} = \varepsilon c^{-1} M_3^0 \tag{12.3}$$

$$\dot{\psi} = \varepsilon \operatorname{cosec}\theta(a\sin\alpha - b\cos\alpha) + \varepsilon K_0 c^{-1} r_0^{-1} - \varepsilon^2 K_0 c^{-1} r_0^{-2}\delta \tag{12.4}$$

$$\dot{\theta} = \varepsilon(a\cos\alpha + b\sin\alpha) \tag{12.5}$$

Here M_i^0 denotes functions obtained from M_i^* as a result of substitution (8)–(10), *i.e.*,

$$M_i^0(a,b,r,\psi,\theta,\alpha,\gamma,t) = M_i^*(P,Q,r,\psi,\theta,\phi,t), \quad i = 1, 2, 3 \tag{13}$$

The system of equations (12) can be written in the form :

$$\dot{x} = \varepsilon F_1(x,y) + \varepsilon^2 F_2(x,y), \; x(0) = x_0, \tag{14.1}$$

$$\dot{y}^1 = \omega_1 + \varepsilon g_1(x,y) + \varepsilon^2 g_2(x,y), \; y^1(0) = y^{10}, \tag{14.2}$$

$$\dot{y}^2 = \omega_2 + \varepsilon h_1(x,y) + \varepsilon^2 h_2(x,y), \; y^2(0) = y^{20}, \tag{14.3}$$

where the vector-function $x = (x^1, x^2, \dots, x^5)$ represents slow variables a, b, δ, ψ, θ, while y^1 and y^2 represent fast variables α, δ. In (4) ω_1, ω_2 are constants phases and equal to $c(A^0)^{-1} r_0$ and $(c - A^0)(A^0)^{-1} r_0$, respectively. The vector-functions F_i, g_i, h_i $(i = 1, 2)$ determine the right hand sides of system equations (12) and construction treatment of approximate system (14) is described in References [7] and [9]. In accordance with this treatment we will change to variables :

$$x = x^* + \varepsilon u_1(x^*,y^*) + \varepsilon^2 u_2(x^*,y^*) + \dots,$$

$$y = y^* + \varepsilon v_1(x^*,y^*) + \varepsilon^2 v_2(x^*,y^*) + \dots,$$

$$y = (y^1, y^2), \qquad x^* = (x^{*1}, \dots, x^{*5})$$

$$y^* = (y^{*1}, y^{*2}), \qquad (g_1, h_1) = z_1$$

Thus system of equations (14) in new variables (x^*, y^*) takes the form :

$$\dot{x}^* = \varepsilon A_1(x^*) + \varepsilon^2 A_2(x^*) + \dots, \tag{15.1}$$

$$\dot{y}^* = \omega + \varepsilon B_1(x^*) + \varepsilon^2 B_2(x^*) + \dots, \quad \omega = (\omega_1, \omega_2) \tag{15.2}$$

It is known [9], that the equations of vector-functions u_1, v_1 have the form :

$$\omega \frac{\partial u_1}{\partial y^*} = F_1(x^*,y^*) - A_1(x^*) \tag{16.1}$$

$$\omega \frac{\partial v_1}{\partial y^*} = z_1(x^*,y^*) - B_1(x^*) \tag{16.2}$$

and through $(\partial F/\partial x)$ we express the differential matrix with respect to fast variables $\| \partial F_i / \partial x_j \|$ $(i, j = 1, \dots, 5)$ functions $A_1(x^*)$, $B_1(x^*)$ are determined in the form :

$$A_1(x^*) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} F_1(x^*,y^*) dy^{*1} dy^{*2} \tag{17.1}$$

$$B_1(x^*) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} z_1(x^*, y^*) dy^{*1} dy^{*2} \tag{17.2}$$

The function $u_2(x^*, y^*)$ is the solution of the following equation

$$\frac{\partial u_2}{\partial y^*}\omega = F_2(x^*, y^*) + \frac{\partial F_1}{\partial x^*}u_1 + \frac{\partial F_1}{\partial y^*}v_1 - \frac{\partial u_1}{\partial x^*}A_1(x^*)$$
$$- \frac{\partial u_1}{\partial y^*}B_1(x^*) - A_2(x^*), \tag{18}$$

where $$A_2(x^*) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\left(F_2 + \frac{\partial F_1}{\partial x^*}u_1 + \frac{\partial F_1}{\partial y^*}v_1 - \frac{\partial u_1}{\partial x^*}A_1 - \frac{\partial u_1}{\partial y^*}B_1\right)dy^{*1}dy^{*2} \tag{19}$$

We determine the first approximation of averaging system of equations of slow variables as :

$$\dot{x}_1^* = \varepsilon A_1(x_1^*), \quad x_1^*(0) = x_{10} \tag{20}$$

and the second approximation for slow variables as :

$$\dot{x}_2^* = \varepsilon A_1(x_2^*) + \varepsilon^2 A_2(x_2^*), \quad x_2^*(0) = x_{20} \tag{21}$$

and the second approximation of system of equations for fast variables as :

$$\dot{y}_2^* = \omega + \varepsilon B_1(x_1(t)), \quad y_2^*(0) = y^0, \quad y^0 = (y^{10}, y^{20}) \tag{22}$$

which is directly integrated to give

$$y_2^*(t) = y^0 + \omega t + \varepsilon\int_0^t B(x_1^*(s))ds \tag{23}$$

It is convenient to introduce an independent variable $\tau = \varepsilon t$ for investigating the second approximation system (21), so the system (21) takes the form :

$$dx_2^*/dt = A_1(x_2^*) + \varepsilon A_2(x_2^*) \tag{24}$$

Thus, the time interval $(0, T/\varepsilon)$ which we considered in the initial system (14), will be $(0, T)$ and is independent of the small parameter ε. We will assume a solution of the system (24) in the form :

$$x_2^*(\tau) = x^{(1)}(\tau) + \varepsilon x^{(2)}(\tau) + 0(\varepsilon^2). \tag{25}$$

Substituting in (14) we get the following system of equations for vector-function $x^i(\tau) = x_i(\tau)$, $(\tau = \varepsilon t, i = 1, 2)$;

$$dx^{(1)}/d\tau = A_1(x^{(1)}), \quad x^{(1)}(0) = x_0; \tag{26}$$

$$dx^{(2)}/d\tau = A_1'(x^{(1)}(\tau))x^{(2)} + A_2(x^{(1)}(\tau)), \quad x^{(2)}(0) = 0 \tag{27}$$

where A_1' denotes the matrix of differential components of vector-function $A_1(x)$ by fast variables.

Denoting by $X(\tau, c)$, the general solution of the first approximation of system (25) :

$$X_\tau' = A_1(X), \quad X(0,c) = c = X_0$$

Thus, the functions $x^{(1)}(\tau)$, $x^{(2)}(\tau)$ will take the following forms

$$x^{(1)}(\tau) = X(\tau, x_0), \quad x^2(\tau) = \phi(\tau)\int_0^\tau \phi^{-1}(\tau_1)\eta(\tau_1)d\tau_1 \tag{28}$$

where $$\phi(\tau) = \left\| \partial X(\tau, c)/\partial c \right\|_{c=x_0},$$

$$\eta(\tau) = A_2(x^{(1)}(\tau)) = A_2(X(\tau, x_0))$$

We define the vector-function

$$\tilde{x}_\varepsilon(t) = x^{(1)}(\varepsilon t) + \varepsilon x^{(2)}(\varepsilon t) + \varepsilon u_1(x^{(1)}(\varepsilon t), y^0 + \omega t$$
$$+ \varepsilon\int_0^t B_1(x^{(1)}(\varepsilon s))ds) \tag{29.1}$$

$$\tilde{y}_\varepsilon(t) = y^\circ + \omega t + \varepsilon\int_0^t B_1(x^{(1)}(\varepsilon s))ds \tag{29.2}$$

Thus, the approximate solutions $\tilde{x}_\varepsilon(t)$, $\tilde{y}_\varepsilon(t)$, are obtained as follows : with the aid of Fourier series, we solve equations (16) and (18). Hence, using formula (19) we construct the vector-function $A_2(x^*)$, and in accordance with (28) we determine the solutions $x^{(1)}(\tau)$, $x^{(2)}(\tau)$ from eqs. (26) and (27). Finally using formula (29), we get the required approximation.

4. The case of variable restoring moment

As an example, restoring moment depends on the components of the angular velocity and further on the angles θ and ϕ. Consider the rotation of a symmetrical rigid body in an electromagnetic field such that the magnetic field of strength D is horizontal and a point charge (e), on the axis of symmetry. Thus the motion of the rigid body is under the force of gravity and Lorentz force which equal $e(\underline{w} \wedge \underline{D})$ [15] where $\underline{w}$ is the vector of the angular velocity of the rigid body. The resultant of restoring moment K, taking into account of the inequalities (2), can be written in the form :

$$K = mgL + eDL'\left[r - \frac{1}{2}r^{-1}(p\sin\phi\sin\theta + q\cos\phi\sin\theta + r\cos\theta)^2\right] \tag{30}$$

where L' is the distance of the position of point charge (e) from the origin.

In the following, we will employ the above technique to consider some specific examples of perturbed motions of a rigid body.

5. The case of linear-dissipative perturbed moments

Let us consider perturbed Lagrange motion with allowance for the moments acting on the rigid body from the environment. We will assume that the perturbing moments M_i $(i = 1, 2, 3)$ are linear and dissipative [16] :

$$M_1 = -\varepsilon^2 I_1 P,\ M_2 = -\varepsilon^2 I_1 Q,\ M_3 = -\varepsilon^2 I_3 r,\ I_1, I_3 > 0 \tag{31}$$

where I_1, I_3 are constant coefficients of proportionality that depend on the properties of the medium and the shape of the body. The first three eqs. of (12) for the considered problem, with the variables $(a, b, \delta, \psi, \theta, \alpha, \gamma)$ take the form :

$$\begin{aligned}\dot a = & -\varepsilon A^{-1} I_1\left(a + \lambda c^{-1} r_0^{-1}\sin\theta\sin\alpha\right)\varepsilon A^{-1}\mu\sin^2\theta\cos\theta[a\sin\alpha\cos\alpha \\ & -b\cos^2\alpha + \lambda c^{-1} r_0^{-1}\sin\theta\cos\alpha]\varepsilon\lambda c^{-1} r_0^{-1}\cos\theta\left(b - \lambda c^{-1} r_0^{-1}\sin\theta\cos\alpha\right) \\ & -\varepsilon c^{-1}\mu\sin^2\theta\cos\theta\sin\alpha(a\cos\alpha + b\sin\alpha) + \varepsilon^2\lambda c^{-1} r_0^{-2}\delta\cos\theta \\ & \times\left(b - 2\lambda c^{-1} r_0^{-1}\sin\theta\cos\alpha\right) - \varepsilon^2\lambda c^{-2} r_0^{-1} I_3\sin\theta\sin\alpha + \frac{1}{2}\varepsilon^2 c^{-2} I_3\mu \\ & \times\left(1 + \sin^2\theta\right)\sin\theta\sin\alpha - \frac{1}{2}\varepsilon^2 A^{-1}\mu r_0^{-1}\sin^3\theta\cos\alpha(a\sin\alpha - b\cos\alpha \\ & +\lambda c^{-1} r_0^{-1}\sin\theta)^2 + \varepsilon^2\lambda c^{-1} A^{-1}\mu r_0^{-2}\delta\sin^3\theta\cos\theta\cos\alpha \\ & +\varepsilon^2 A^{-1} I_1\lambda c^{-1} r_0^{-2}\delta\sin\theta\sin\alpha,\end{aligned} \tag{32.1}$$

$$\begin{aligned}\dot b = & -\varepsilon A^{-1} I_1\left(b - \lambda c^{-1} r_0^{-1}\sin\theta\cos\alpha\right) + \varepsilon A^{-1}\mu\sin^2\theta\cos\theta\,(a\sin^2\alpha \\ & -b\sin\alpha\cos\alpha + \lambda c^{-1} r_0^{-1}\sin\theta\sin\alpha) + \varepsilon\lambda c^{-1} r_0^{-1}\cos\theta \\ & \times\left(a + \lambda c^{-1} r_0^{-1}\sin\theta\sin\alpha\right) + \varepsilon c^{-1}\mu\sin^2\theta\cos\theta\cos\alpha\,(a\cos\alpha + b\sin\alpha) \\ & -\varepsilon^2\lambda c^{-1} r_0^{-2}\delta\cos\theta\left(a + 2\lambda c^{-1} r_0^{-1}\sin\theta\sin\alpha\right) + \varepsilon^2\lambda c^{-2} r_0^{-1} I_3\sin\theta\cos\alpha \\ & -\frac{1}{2}\varepsilon^2 c^{-2} I_3\mu(1 + \sin^2\theta)\sin\theta\cos\alpha + \frac{1}{2}\varepsilon A^{-1}\mu r_0^{-1}\sin^3\theta\sin\alpha(a\sin\alpha \\ & -b\cos\alpha + \lambda c^{-1} r_0^{-1}\sin\theta)^2 - \varepsilon^2\lambda c^{-1} A^{-1}\mu r_0^{-2}\delta\sin^3\theta\cos\theta\sin\alpha \\ & -\varepsilon^2 A^{-1} I_1\lambda c^{-1} r_0^{-2}\delta\sin\theta\cos\alpha,\end{aligned} \tag{32.2}$$

$$\dot\delta = -\varepsilon c^{-1} I_3 r_0 - \varepsilon^2 c^{-1} I_3\delta, \tag{32.3}$$

where

$$\begin{aligned}M_1^0\cos\gamma + M_2^0\sin\gamma = & -\varepsilon I_1 A^{-1}\left[a + \lambda c^{-1} r_0^{-1}\sin\theta\sin\alpha\right] \\ & +\varepsilon^2 A^{-1} I_1\lambda c^{-1} r_0^{-2}\delta\sin\theta\sin\alpha\end{aligned} \tag{33.1}$$

$$\begin{aligned}M_1^0\sin\gamma - M_2^0\cos\gamma = & -\varepsilon I_1 A^{-1}\left[b - \lambda c^{-1} r_0^{-1}\sin\theta\cos\alpha\right] \\ & -\varepsilon^2 A^{-1} I_1\lambda c^{-1} r_0^{-2}\delta\sin\theta\cos\alpha\end{aligned} \tag{33.2}$$

The other four equations are the same as in (12).

Applying the last described average method to the system of equations (33), the components of vector-functions A_1 and B_1 which are defined by the formula (17) take the form :

$$A^{(1)} = -A^{-1}la - \lambda c^{-1}r^{-1}b\cos\theta - \frac{1}{2}c^{-1}\mu b\sin^2\theta\cos\theta + \frac{1}{2}A^{-1}\mu b\sin^2\theta\cos\theta, \quad (34.1)$$

$$A_1^{(2)} = -A^{-1}l_1 b + \lambda c^{-1}r_0^{-1}a\cos\theta + \frac{1}{2}c^{-1}\mu a\sin^2\theta\cos\theta + \frac{1}{2}A^{-1}\mu a\sin^2\theta\cos\theta, \quad (34.2)$$

$$A_1^{(3)} = -c^{-1}l_3 r_0, \quad A_1^{(4)} = \lambda c^{-1}r_0^{-1}, \quad A_1^{(5)} = 0, \quad (34.3)$$

$$B_1^{(1)} = cA^{-1}\delta - \lambda c^{-1}r_0^{-1}\cos\theta, \quad (34.4)$$

$$B_1^{(2)} = (c-A)A^{-1}\delta. \quad (34.5)$$

The components $u_1^{(4)}$ and $u_1^{(5)}$ of vector-function $u_1 = \{u_1^{(i)}\}, (i = 1, 2, \ldots\ldots, 5)$ take the form :

$$u_1^{(4)} = -c^{-1}Ar_0^{-1}\operatorname{cosec}\theta(a\cos\alpha + b\sin\alpha) \quad (35.1)$$

$$u_1^{(5)} = c^{-1}Ar_0^{-1}(a\sin\alpha - b\cos\alpha) \quad (35.2)$$

The expressions of $M_1^0\cos\gamma + M_2^0\sin\gamma$ and $M_1^0\sin\gamma - M_2^0\cos\gamma$, as obtained from eqs. (33) are independent of the variable γ and right hand side of these equations contain only one fast variable α. This fact is maintained in [17] as enough condition for possibility of averaging the equations of motion only by the angle of nutation.

Using formula (19), the components of vector-function $A_2(x^*)$, take the form :

$$A_2^{(1)} = c^{-1}r_0^{-1}b\Big[\lambda r_0^{-1}\delta\cos\theta + \frac{1}{2}A^{-1}\mu r_0^{-1}\lambda\sin^4\theta - \frac{1}{8}c^{-1}\mu^2\sin^4\theta\cos^2\theta \times(Ac^{-1}-1) - \frac{1}{2}c^{-2}Ar_0^{-2}\big(\lambda^2\cos^2\theta - \lambda^2\sin^2\theta + 2r_0\lambda\mu\sin^2\theta\cos^2\theta\big) + \frac{1}{2}\mu c^{-1}r_0^{-1}\big(r_0\mu\sin^4\theta\cos^2\theta + 3\lambda\sin^2\theta\cos^2\theta - \lambda\sin^4\theta\big) - \frac{1}{2}\lambda^2 Ac^{-2}r_0^{-2}\cos^2\theta + \frac{1}{2}\mu\lambda c^{-1}r_0^{-1}\sin^2\theta\cos^2\theta\Big] - \frac{1}{2}l_1 ac^{-2}r_0^{-2} \times[2\lambda\cos\theta + r_0\mu\sin^2\theta\cos\theta], \quad (36.1)$$

$$A_2^{(2)} = c^{-1}r_0^{-1}a\Big[-\lambda r_0^{-1}\delta\cos\theta + \frac{1}{2}A^{-1}\mu r_0^{-1}\lambda\sin^4\theta + \frac{1}{8}c^{-1}\mu^2\sin^4\theta\cos^2\theta \times(Ac^{-1}-1) - \frac{1}{2}c^{-2}Ar_0^{-2}\big(\lambda^2\cos^2\theta - \lambda^2\sin^2\theta + 2r_0\lambda\mu\sin^2\theta\cos^2\theta\big) + \frac{1}{2}\mu c^{-1}r_0^{-1}\big(r_0\mu\sin^4\theta\cos^2\theta + 3\lambda\sin^2\theta\cos^2\theta - \lambda\sin^4\theta\big) + \frac{1}{2}\lambda^2 Ac^{-2}r_0^{-2}\cos^2\theta + \frac{1}{2}\mu\lambda c^{-1}r_0^{-1}\sin^2\theta\cos^2\theta\Big] - \frac{1}{2}l_1 bc^{-2}r_0^{-2} \times[2\lambda\cos\theta + r_0\mu\sin^2\theta\cos\theta], \quad (36.2)$$

$$A_2^{(3)} = -c^{-1} I_3 \delta, \tag{36.3}$$

$$A_2^{(4)} = -\lambda c^{-1} r_0^{-2} \delta + A\lambda^2 c^{-3} r_0^{-3} \cos\theta, \tag{36.4}$$

$$A_2^{(5)} = \lambda I_1 c^{-2} r_0^{-2} \sin\theta \tag{36.5}$$

Hence, the solution of the averaged system equations (20) in the first approximation for slow and fast variables takes the form :

$$a^{(1)} = \exp(-\varepsilon A^{-1} I_1 t)(a^0 \cos\eta t - b^0 \sin\eta t), \tag{37.1}$$

$$b^{(1)} = \exp(-\varepsilon A^{-1} I_1 t)(b^0 \cos\eta t + a^0 \sin\eta t), \tag{37.2}$$

$$\delta^{(1)} = -\varepsilon c^{-1} I_3 r_0 t, \tag{37.3}$$

$$\psi^{(1)} = \varepsilon\lambda c^{-1} r_0^{-1} t + \psi_0, \tag{37.4}$$

$$\theta^{(1)} = \theta_0, \tag{37.5}$$

$$\alpha^{(1)} = cA^{-1} r_0 t - \varepsilon\lambda c^{-1} r_0^{-1} \cos\theta_0 t - \frac{1}{2}\varepsilon^2 A^{-1} I_3 r_0 t^2 + \phi_0, \tag{37.6}$$

$$\gamma^{(1)} = n_0 t - \frac{1}{2}\varepsilon^2 (c - A) A^{-1} c^{-1} I_3 r_0 t^2, \tag{37.7}$$

where $$\eta = \varepsilon\left[\varepsilon\lambda c^{-1} r_0^{-1} \cos\theta - \frac{1}{2} c^{-1} \mu \sin^2\theta \cos\theta\right]^2 - \varepsilon\left[\frac{1}{2}\mu A^{-1} \sin^2\theta \cos\theta\right]^2,$$

and the quantities a^0, b^0, n_0 are determined in accordance with (6); ψ_0, θ_0, ϕ_0 are constants and equal to the initial values of the corresponding angles at $t = 0$.

On the basis of a given formulae (29), we can construct the components of the functions $x_\varepsilon(t)$ and $\psi_\varepsilon(t)$ for the variables ψ, θ to take the form :

$$\begin{aligned}\tilde{\psi}_\varepsilon(t) = {} & \psi_0 + \varepsilon\big[\lambda c^{-1} r_0^{-1} t - c^{-1} r_0^{-1} A \operatorname{cosec}\theta \exp(-\varepsilon A^{-1} I_1 t) \\ & \times \left[a^0 \cos(\alpha^{(1)} - \eta t) + b^0 \sin(\alpha^{(1)} - \eta t)\right] + \varepsilon^2\big(-\lambda c^{-1} r_0^{-2} \delta^{(1)} \\ & + A\lambda^3 c^{-3} r_0^{-3} \cos\theta\big)\big],\end{aligned} \tag{38.1}$$

$$\begin{aligned}\bar{\theta}_\varepsilon(t) = {} & \theta_0 + \varepsilon c^{-1} r_0^{-1} A_0 \exp(-\varepsilon A^{-1} I_1 t)\left[a^0 \sin(\eta t - \alpha^{(1)})\right. \\ & \left. + b^0 \cos(\eta t - \alpha^{(1)})\right].\end{aligned} \tag{38.2}$$

Thus, we have constructed the solution of the second approximation system for the precession and nutation angles. The above equations are solved numerically. Figure 1 shows the behaviour of $\psi_\varepsilon(t)$ at given initial $\psi_0 = \pi/4$ for different values of magnetic field. From this figure, it is observed that the rotation of ψ is steady at the beginning for a short time and hence will be oscillatory. The intervals of steady rotation and oscillations are increasing with increasing of time. Also, we note that the behaviour of $\psi_\varepsilon(t)$

at various values of magnetic field is similar but there exist dilatation in time Figure 2 shows the behaviour of angle $\theta_\varepsilon(t)$ at initial $\theta_0 = \pi/4$ and for different values of magnetic

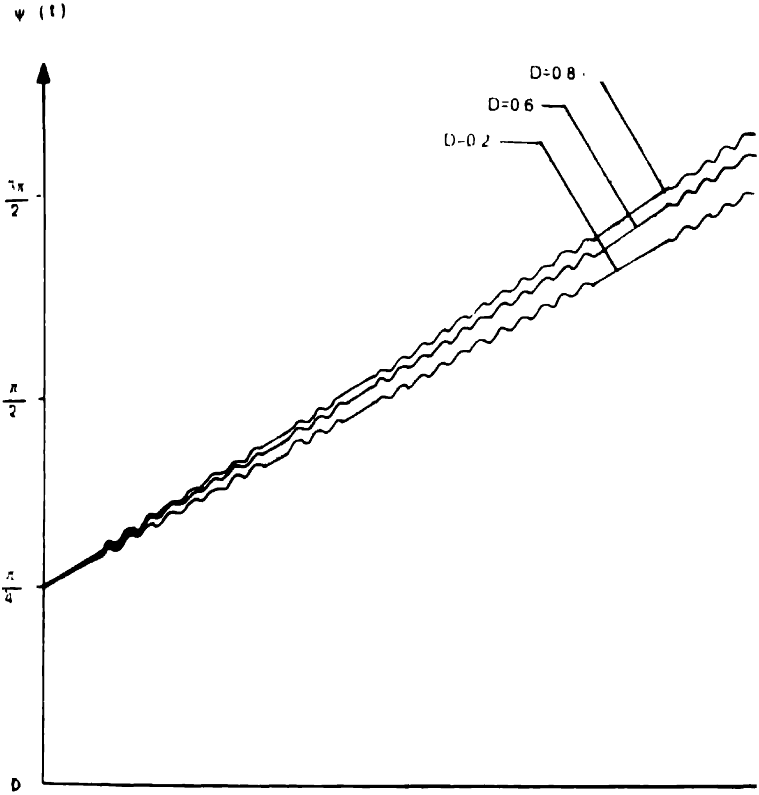

Figure 1. Shows the behaviour of $\psi(t)$ for a given initial value $\psi_0 = \pi/4$ and for different values of magnetic field ($\varepsilon = 0.01$, $D = -0.2, 0.6, 0.8$)

field, we note that all the motions are oscillatory. Also, we note that the period of time is increasing with the decrease of the magnetic field.

6. The case of perturbed moments producing from a cavity filled with high viscous fluid

Let us consider perturbed rotational motion of a symmetrical rigid body containing a cavity filled with incompressible viscous fluid. The components of the vector of perturbing moment in this case, take the form [18] :

$$M_1 = \rho A^{-2} c^{-1} P \nu^{-1} \Big[c^2 (A - c) p r^2 + K c (c - A) r \sin\theta \sin\phi + K A c p \cos\theta \Big], \tag{39.1}$$

$$M_2 = \rho A^{-2} c^{-1} P \nu^{-1} \left[c^2 (A - c) q r^2 + Kc(c - A) r \sin\theta \cos\phi + KAcq \cos\theta \right] \quad (39.2)$$

$$M_3 = \rho A^{-2} c^{-1} P \nu^{-1} \left[Ac(c - A)(p^2 + q^2) r - KAc \sin\theta (p \sin\phi + q \cos\phi) \right] \quad (39.3)$$

Here, ρ and ν are the density and the kinematic viscosity of the fluid. The constant symmetric tensor P depends only on the form and dimensions of the cavity. Since in the considerable case of heigh viscous fluid, Reynold's number is small, thus we can replace ν^{-1} by the small parameter ε. The first three equations of (12) in this case, take the form :

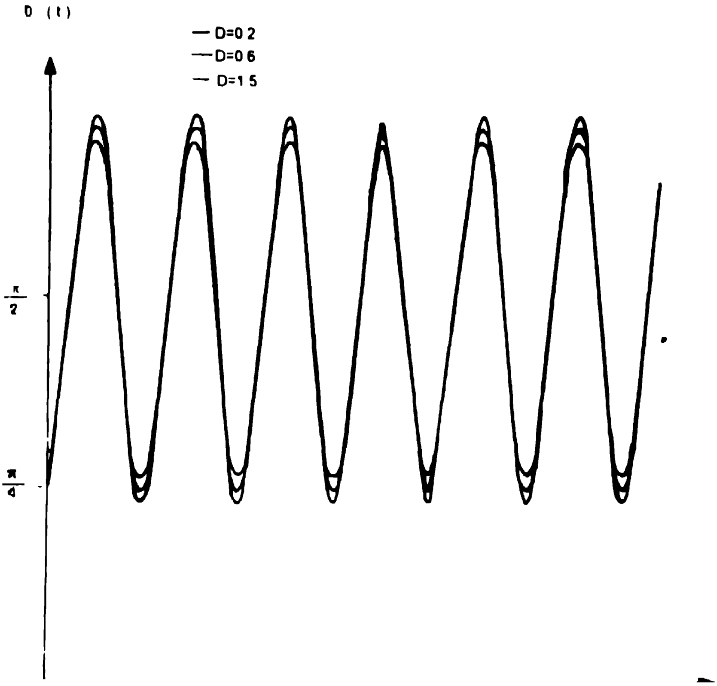

Figure 2. Shows the behaviour of $\theta(t)$ for a given initial value $\theta_0 = \pi/4$ and for different values of magnetic field ($\varepsilon = 0.01$, $D = 0.2, 0.8, 1.5$).

$$\dot{a} = \varepsilon A^{-3} \rho P c (A - c) r_0^{-2} a - \varepsilon \mu A^{-1} \sin^2\theta \cos\theta \left[a \sin\alpha \cos\alpha - b \cos^2\alpha + \lambda c^{-1} r_0^{-1} \sin\theta \cos\alpha \right] - \varepsilon \lambda c^{-1} r_0^{-1} \cos\theta \left(b - \lambda c^{-1} r_0^{-1} \sin\theta \cos\alpha \right) - \varepsilon \mu c^{-1} \sin^2\theta \cos\theta \sin\alpha (a \cos\alpha + b \sin\alpha) + 2\varepsilon^2 A^{-3} \delta r_0 P \rho (A - c) a$$

$$- \varepsilon^2 \mu \rho P r_0 A^{-3}(c-A)\sin^2\theta\cos\theta\,(a\sin^2\alpha - b\sin\alpha\cos\alpha$$

$$+ \lambda c^{-1} r_0^{-1}\sin\theta\sin\alpha) + \varepsilon^2\lambda c^{-1} r_0^{-2}\delta\cos\theta(b - 2\lambda c^{-1} r_0^{-1}\sin\theta\cos\alpha)$$

$$- \frac{1}{2}\varepsilon^2 A^{-1} r_0^{-1}\sin^3\theta\cos\alpha\left(a\sin\alpha - b\cos\alpha + \lambda c^{-1} r_0^{-1}\sin\theta\right)^2$$

$$+ \varepsilon^2\lambda c^{-1} A^{-1}\mu r^{-2}\delta\sin^3\theta\cos\theta\cos\alpha \tag{40.1}$$

$$\dot{b} = \varepsilon A^{-3}\rho P c(A-c) r_0^{-2} b - \varepsilon\mu A^{-1}\sin^2\theta\cos\theta\left[a\sin^2\alpha - b\sin\alpha\cos\alpha\right.$$

$$\left.+ \lambda c^{-1} r_0^{-1}\sin\theta\sin\alpha\right] + \varepsilon\lambda c^{-1} r_0^{-1}\cos\theta\left(b + \lambda c^{-1} r_0^{-1}\sin\theta\sin\alpha\right)$$

$$+ \varepsilon\mu c^{-1}\sin^2\theta\cos\theta\cos\alpha(a\cos\alpha + b\sin\alpha) + 2\varepsilon^2 A^{-3}\delta r_0 P\rho(A-c)b$$

$$+ \varepsilon^2\mu\rho P r_0 A^{-3}(c-A)\sin^2\theta\cos\theta\,(a\sin\alpha\cos\alpha - b\cos^2\alpha$$

$$+ \lambda c^{-1} r_0^{-1}\sin\theta\cos\alpha) - \varepsilon^2\lambda c^{-1} r_0^{-2}\delta\cos\theta(a + 2\lambda c^{-1} r_0^{-1}\sin\theta\sin\alpha)$$

$$+ \frac{1}{2}\varepsilon^2 c^{-1} A^{-1}\mu r_0^{-2}\delta\sin^3\theta\sin\alpha\left(a\sin\alpha - b\cos\alpha + \lambda c^{-1} r_0^{-1}\sin\theta\right)^2$$

$$- \varepsilon^2\mu A^{-1}\delta c^{-1}\lambda r_0^{-2}\sin^3\theta\cos\theta\sin\alpha \tag{40.2}$$

$$\dot{\delta} = 0 \tag{40.3}$$

The other four equations will be the same as in (12).

Utilizing the last described average method for the system of eqs. (40), the components of vector-functions A_1 and B_1 which are defined by the formula (17), take the form .

$$A_1^{(1)} = A^{-3}\rho P c(A-c) r_0^2 a - \lambda c^{-1} r_0^{-1} b\cos\theta - \frac{1}{2} b c^{-1}\mu\sin^2\theta\cos\theta$$

$$+ \frac{1}{2}\mu b A^{-1}\sin^2\theta\cos\theta, \tag{41.1}$$

$$A_1^{(2)} = A^{-3}\rho P c(A-c) r_0^2 b + \lambda c^{-1} r_0^{-1} a\cos\theta + \frac{1}{2} a c^{-1}\mu\sin^2\theta\cos\theta$$

$$+ \frac{1}{2}\mu a A^{-1}\sin^2\theta\cos\theta, \tag{41.2}$$

$$A_1^{(3)} = 0, \tag{41.3}$$

$$A_1^{(4)} = \lambda c^{-1} r_0^{-1}, \tag{41.4}$$

$$A_1^{(5)} = 0, \tag{41.5}$$

$$B_1^{(1)} = c A^{-1}\delta - \lambda c^{-1} r_0^{-1}\cos\theta, \tag{41.6}$$

$$B_1^{(2)} = (c-A) A^{-1}\delta. \tag{41.7}$$

The components $u_1^{(4)}$ and $u_1^{(5)}$ of vector-function $u_1 = \{u_1^{(i)}\}$, $(i = 1, 2, \ldots, 5)$ take the form :

$$u_1^{(4)} = -Ac^{-1}r_0^{-1}\operatorname{cosec}\theta(a\cos\alpha + b\sin\alpha) \tag{42.1}$$

$$u_1^{(5)} = Ac^{-1}r_0^{-1}(a\sin\alpha - b\cos\alpha) \tag{42.2}$$

Here, the expressions for $M_1^{(0)}\cos\gamma + M_2^{(0)}\sin\gamma$ and $M_1^{(0)}\sin\gamma - M_2^{(0)}\cos\gamma$, depend only on one fast variable α, which allow averaging the equations of motion only by the angle of nutation θ, as mentioned above.

The components of vector-function $A_2(x^*)$, using formula (19), take the form :

$$\begin{aligned} A_2^{(1)} = {} & \frac{1}{2}\rho PA^{-3}ar_0(A-c)\left[4\delta c + \mu\sin^2\theta\cos\theta\right] + bc^{-1}r_0^{-1}\Big\{\lambda\delta r_0^{-1}\cos\theta \\ & + \frac{1}{2}A^{-1}\mu\lambda r_0^{-1}\sin^4\theta - \frac{1}{8}c^{-1}\mu^2\sin^4\theta\cos^2\theta(c^{-1}A-1) \\ & - \frac{1}{2}c^{-2}r_0^{-2}A\left(\lambda^2\cos^2\theta - \lambda^2\sin^2\theta + 2r_0\lambda\mu\sin^2\theta\cos^2\theta\right) \\ & + \frac{1}{2}\mu c^{-1}r_0^{-1} \times \left(r_0\mu\sin^4\theta\cos^2\theta + 3\lambda\sin^2\theta\cos^2\theta - \lambda\sin^4\theta\right) \\ & - \frac{1}{2}\lambda^2Ac^{-2}r_0^{-2}\cos^2\theta + \frac{1}{2}\mu\lambda c^{-1}r_0^{-1}\sin^2\theta\cos^2\theta\Big\}, \end{aligned} \tag{43.1}$$

$$\begin{aligned} A_2^{(2)} = {} & \frac{1}{2}\rho PA^{-3}br_0(A-c)\left[4\delta c + \mu\sin^2\theta\cos\theta\right] - ac^{-1}r_0^{-1}\Big\{\lambda\delta r_0^{-1}\cos\theta \\ & + \frac{1}{2}A^{-1}\mu r_0^{-1}\sin^4\theta - \frac{1}{8}c^{-1}\mu^2\sin^4\theta\cos^2\theta(c^{-1}A-1) \\ & - \frac{1}{2}c^{-2}r_0^{-2}A\left(\lambda^2\cos^2\theta - \lambda^2\sin^2\theta + 2r_0\lambda\mu\sin^2\theta\cos^2\theta\right) \\ & - \frac{1}{2}\mu c^{-1}r_0^{-1}\left(r_0\mu\sin^4\theta\cos^2\theta + 3\lambda\sin^2\theta\cos^2\theta - \lambda\sin^4\theta\right) \\ & - \frac{1}{2}\lambda^2Ac^{-2}r_0^{-2}\cos^2\theta + \frac{1}{2}\mu\lambda c^{-1}r_0^{-1}\sin^2\theta\cos^2\theta\Big\}, \end{aligned} \tag{43.2}$$

$$A_2^{(3)} = 0, \tag{43.3}$$

$$A_2^{(4)} = -\lambda\delta c^{-1}r_0^{-2} + A\lambda^2c^{-3}r_0^{-3}\cos\theta, \tag{43.4}$$

$$A_2^{(5)} = 0, \tag{43.5}$$

The averaged system equations of the first approximation (20) are determined, for slow and fast variables to take the form :

$$a^{(1)} = \exp(\varepsilon A^{-3}\rho Pc(A-c)r_0^2t)(a^0\cos\eta t - b^0\sin\eta t), \tag{44.1}$$

$$b^{(1)} = \exp(\varepsilon A^{-3}\rho Pc(A-c)r_0^2t)(b^0\cos\eta t + a^0\sin\eta t), \tag{44.2}$$

$$\delta^{(1)} = \delta^{(0)}, \tag{44.3}$$

$$\psi^{(1)} = \varepsilon\lambda c^{-1} r_0^{-1} t + \psi_0 \tag{44.4}$$

$$\theta^{(1)} = \theta_0 \tag{44.5}$$

$$\alpha^{(1)} = cA^{1} r_0 t - \varepsilon\lambda c^{-1} r_0^{-1} \cos\theta_0 t - \frac{1}{2}\varepsilon^2 A^{-1} t^2 + \phi_0 \tag{44.6}$$

$$\delta^{(1)} = n_0 t - \frac{1}{2}\varepsilon^2 (c-A) A^{-1} c^{-1} t^2 \tag{44.7}$$

where $$\eta = \varepsilon\left[\lambda c^{-1} r_0^{-1} \cos\theta_0 + \frac{1}{2} c^{-1}\mu \sin^2\theta_0 \cos\theta_0 - \frac{3}{4}\mu A^{-1} \sin^2\theta_0 \cos\theta_0\right]^2$$

The quantities a^0, b^0, n_0 are determined in accordance with (6); ψ_0, θ_0, ϕ_0 —are constants and equal to the initial values of the corresponding angles.

On the basis of the given formulae, we can construct the components of the function x_ε^v for the variables ψ, θ to take the form :

$$\begin{aligned} \psi_\varepsilon^v(t) &= \psi_0 + \varepsilon\{\lambda c^{-1} r_0^{-1} t - c^{-1} r_0^{-1} A \operatorname{cosec}\theta_0 \exp(\varepsilon A^{-3}\rho P c(A-c) r_0^2 t) \\ &\times \{a^0 \cos(\alpha^{(1)} - \eta t) + b^0 \sin(\alpha^{(1)} - \eta t)\} + \varepsilon^2(-\lambda c^{-1} r_0^{-2} \delta^{(0)} \\ &+ A\lambda^2 c^{-3} r_0^{-3} \cos\theta_0), \end{aligned} \tag{45.1}$$

$$\begin{aligned} \theta_\varepsilon^v(t) &= \theta_0 + \varepsilon c^{-1} r_0^{-1} A \exp(\varepsilon A^{-3}\rho P c(A-c) r_0^2 t)\{a^0 \sin(\alpha^{(1)} - \eta t) \\ &- b^0 \cos(\alpha^{(1)} - \eta t)\}. \end{aligned} \tag{45.2}$$

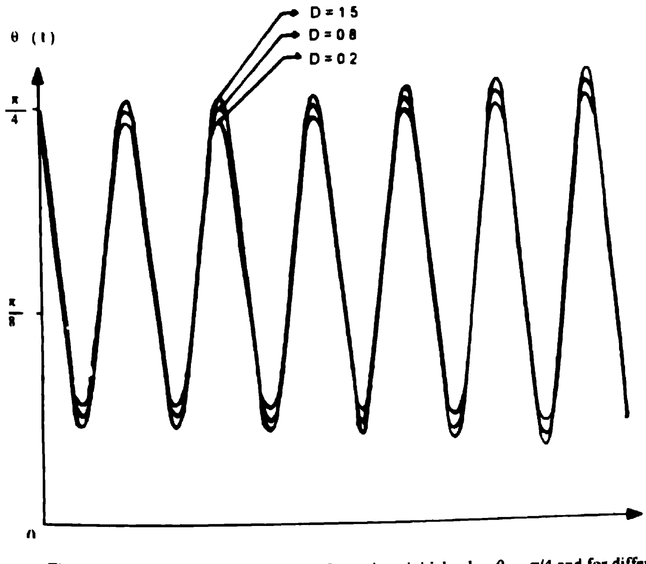

Figure 3. Shows the behaviour of $\theta(t)$ for a given initial value $\theta_0 = \pi/4$ and for different values of magnetic field ($\varepsilon = 0.001$, $D = 0.2, 0.8, 1.5$).

Now, we will solve the above equations numerically to show the dependence of θ_ε and ψ_ε on the value of magnetic field with time. From Figure 3 it is noticed that $\theta_\varepsilon(t)$ is a periodic function with time and it is increasing with the increase of the magnetic field. From Figure 4 we notice that $\psi_\varepsilon(t)$ increases at decreasing of magnetic field with time.

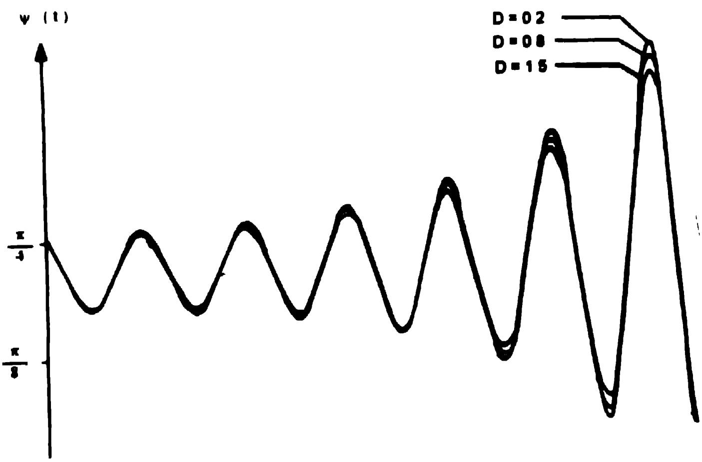

Figure 4. Shows the behaviour of $\psi(t)$ for a given initial value $\psi_0 = \pi/4$ and for different values of magnetic field ($\varepsilon = 0.01$, $D = 0.2, 0.8, 1.5$)

The variation of ψ_ε has no regularity and the effect of magnetic field is very small at large time and this effect is negligible at small time.

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